

Necessary and Sufficient Conditions for Distinguishability of Linear Control Systems *

Hongwei Lou[†]

Abstract. Distinguishability takes a crucial rule in studying observability of hybrid system such as switched system. Recently, for two linear systems, Lou and Si gave a condition not only necessary but also sufficient to the distinguishability of linear systems. However, the condition is not easy enough to verify. This paper will give a new equivalent condition which is relatively easy to verify.

Key words and phrases. distinguishability, linear control systems, necessary and sufficient condition.

AMS subject classifications. 34H05, 93B99

1 Introduction

Consider a switched system composed by two time-invariant subsystems ($i = 1, 2$):

$$S_i : \begin{cases} \frac{dx}{dt} = A_i x(t) + B_i u(t), \\ y(t) = C_i x(t) + G_i u(t), \end{cases} \quad (1.1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^k$. Naturally,

$$A_i \in \mathbb{R}^{n \times n}, \quad B_i \in \mathbb{R}^{n \times m}, \quad C_i \in \mathbb{R}^{k \times n}, \quad G_i \in \mathbb{R}^{k \times m}. \quad (1.2)$$

Switched system is an important case of hybrid systems. When we consider the observability of switched system composed by time-invariant subsystems such as system (1.1), distinguishability takes a crucial rule (see [6], [8]). Among the references about observability/distinguishability of hybrid system, we would like to refer the readers to the papers [1], [2], [3], [4], [5], [7], [9], [10], [11], [13] and [14].

In [14], the authors got a necessary and sufficient condition for distinguishability of two linear automation systems (i.e. $B_1 = B_2 = 0$, $G_1 = G_2 = 0$). However, as pointed out by the authors

*This work was supported in part by NSFC (No. 61074047 and 10831007), and 973 Program (No. 2011CB808002).

[†]School of Mathematical Sciences, and LMNS, Fudan University, Shanghai 200433, China (Email: hwlou@fudan.edu.cn).

of [14], for non-automation system, the input plays a crucial role and the distinguishability of two linear systems becomes very difficult. Recently, in [8], the authors gave a definition of distinguishability for linear non-automation systems (see Definition 1.1 below), and yielded a necessary and sufficient condition for distinguishability of two linear systems.

Definition 1.1 (distinguishability) *Systems S_1 and S_2 are said to be **distinguishable** on $[0, T]$, if for any non-zero*

$$(x_{10}, x_{20}, u(\cdot)) \in \mathbf{R}^n \times \mathbf{R}^n \times L^1(0, T; \mathbf{R}^m),$$

the corresponding outputs $y_1(\cdot)$ and $y_2(\cdot)$ can not be identical to each other on $[0, T]$.

To study the distinguishability of two systems, some auxiliary concepts of distinguishability was also introduced in [8]:

Definition 1.2 *Given $T > 0$. Let $\mathcal{U} \subseteq L^1(0, T; \mathbf{R}^m)$ be a function space. We say that S_1 and S_2 are \mathcal{U} **input distinguishable** on $[0, T]$ if for any non-zero*

$$(x_{10}, x_{20}, u(\cdot)) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathcal{U},$$

the outputs $y_1(\cdot)$ and $y_2(\cdot)$ can not be identical to each other on $[0, T]$.

Epecially, when \mathcal{U} is the set of polynomial function class, the set of analytic function class and the set of smooth function class $C^\infty([0, T]; \mathbf{R}^m)$, then the corresponding distinguishability is called “polynomial input distinguishability”, “analytic input distinguishability” and “smooth input distinguishability”, etc.

Denote

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, C = (C_1 \quad -C_2), G = G_1 - G_2 \quad (1.3)$$

and

$$X_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}, Y(\cdot) = y_1(\cdot) - y_2(\cdot). \quad (1.4)$$

Then the distinguishability of S_1 and S_2 on $[0, T]$ is equivalent to that for the following system:

$$S : \begin{cases} \frac{dX}{dt} = AX(t) + Bu(t), \\ X(0) = X_0; \\ Y(t) = CX(t) + Gu(t), \end{cases} \quad (1.5)$$

$(X_0, u(\cdot)) \neq 0$ implies $Y(\cdot) \not\equiv 0$ on $[0, T]$.

It was proved in [8] that

Theorem 1.3 *The distinguishability of S_1 and S_2 on $[0, T]$ is equivalent to that S_1 and S_2 are analytic input distinguishable. Moreover, it is equivalent to that the infinite dimensional equation*

$$\mathcal{M}\beta \equiv \begin{pmatrix} C & G & 0 & 0 & 0 & \dots \\ CA & CB & G & 0 & 0 & \dots \\ CA^2 & CAB & CB & G & 0 & \dots \\ CA^3 & CA^2B & CAB & CB & G & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \vdots \end{pmatrix} = 0. \quad (1.6)$$

admits only trivial solution. Consequently, it is independent of T .

The disadvantage of Theorem 1.3 is that whether equation (1.6) admitting only trivial solution is not easy to verify. In this paper, we will seek for an equivalent condition which can be verified easier.

2 Properties of Differential Operator D and Laplace Transform \mathcal{L}

We recall the notions of differential operator D and Laplace transform \mathcal{L} and list some useful properties of them. For the cause of notation simplicity, in this section, matrices A, B, C, G and integers n, m, k and etc., can be different from that in other sections.

Denote by \mathbb{C} the space of complex numbers. Let

$$P(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0$$

be a polynomial, where $a_k \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. For smooth vector valued function $f : [0, T] \rightarrow \mathbb{C}^k$, define

$$P(D)f(t) = a_n \frac{d^n f(t)}{dt^n} + a_{n-1} \frac{d^{n-1} f(t)}{dt^{n-1}} + \dots + a_1 f'(t) + a_0 f(t). \quad (2.1)$$

It is well known that

Lemma 2.1 *Let $f, g : [0, T] \rightarrow \mathbb{C}^n$ be two smooth vector valued functions on $[0, T]$, $P(\cdot), Q(\cdot)$ are two polynomials, $\alpha, \beta \in \mathbb{C}$ and $\lambda \in \mathbb{C}$ are two complex constants. Then*

$$\left(\alpha P(D) + \beta Q(D) \right) f(t) = \alpha P(D)f(t) + \beta Q(D)f(t), \quad (2.2)$$

$$P(D) \left(\alpha f(t) + \beta g(t) \right) = \alpha P(D)f(t) + \beta P(D)g(t), \quad (2.3)$$

$$\left(P(D)Q(D) \right) f(t) = P(D) \left(Q(D)f(t) \right) = Q(D) \left(P(D)f(t) \right), \quad (2.4)$$

$$P(D) \left(e^{\lambda t} f(t) \right) = e^{\lambda t} P(D + \lambda) f(t). \quad (2.5)$$

For a function $f(\cdot) \in L^1_{loc}[0, +\infty)$, the Laplace transform of $f(\cdot)$ is defined by

$$F(s) \equiv \mathcal{L}(f(\cdot))(s) \triangleq \int_0^{+\infty} e^{-st} f(t) dt, \quad s > 0.$$

It is well known that Laplace transform can be defined for many functions and even for generalized functions such as δ function. If there exist $M_1, M_2 > 0$ such that

$$|f(t)| \leq M_1 e^{M_2 t}, \quad \forall t > 0,$$

then $\mathcal{L}(f(\cdot))(s)$ is well defined for $s \in (M_2, +\infty)$. Moreover, $f(\cdot)$ has the form

$$f(t) = e^{\lambda_1 t} P_1(t) + e^{\lambda_2 t} P_2(t) + \dots + e^{\lambda_n t} P_n(t)$$

with $\lambda_k \in \mathbb{C}$ and $P_k(\cdot)$ being polynomial ($k = 1, 2, \dots$) if and only if $\mathcal{L}(f)$ is a proper rational function.

3 Main Results

Now we consider the necessary and sufficient conditions for distinguishability. Let A, B, C, G be defined as in §1. By the discussions of [8], we know that if S_1 and S_2 are not distinguishable, then they are not analytic input distinguishable. More precisely, there exists a pair $(X_0, u(\cdot))$ such that

$$(X_0, u(\cdot)) \neq 0, \quad (3.1)$$

$$C e^{At} x_0 + C \int_0^t e^{A(t-s)} B u(s) ds + G u(t) = 0, \quad \forall t \geq 0, \quad (3.2)$$

and

$$u(t) = \sum_{j=0}^{\infty} \frac{\alpha_j}{j!} t^j, \quad t \in [0, +\infty), \quad (3.3)$$

with

$$|\alpha_j| \leq M^{j+1}, \quad \forall j = 0, 1, \dots \quad (3.4)$$

for some $M > 0$.

One can see that if $u(\cdot)$ satisfies (3.3)—(3.4), then

$$|u(t)| \leq M e^{Mt}, \quad \forall t \geq 0$$

and therefore $\mathcal{L}(u(\cdot))(s)$ can be defined for any $s > M$.

A crucial property we will prove in the following is that

Lemma 3.1 *If S_1 and S_2 are not distinguishable, then we can find a pair $(\bar{X}_0, \bar{u}(\cdot))$ satisfying (3.1)—(3.2) with*

$$\bar{u}(\cdot) = e^{\lambda_1 t} P_1(t) + e^{\lambda_2 t} P_2(t) + \dots + e^{\lambda_q t} P_q(t), \quad (3.5)$$

where $\lambda_i \in \mathbb{C}$ and $P_i(\cdot)$ are vector-valued polynomials ($i = 1, 2, \dots, q$).

Proof. Since S_1 and S_2 are not distinguishable, by the results of [8], there exists a pair $(X_0, u(\cdot))$ satisfies (3.1)—(3.4). Denote

$$\Phi(s) = \mathcal{L}\left(Ce^{A\cdot}\right)(s), \quad \Psi(s) = \Phi(s)B, \quad U(s) = \mathcal{L}\left(u(\cdot)\right)(s).$$

Then every element in matrices Φ and $\Psi + G$ are rational functions.

Consider the Laplace transform of (3.2), we have

$$\Phi(s)X_0 + (\Psi(s) + G)U(s) = 0. \quad (3.6)$$

Let

$$r = \text{rank} \left(\Psi(s_0) + G \right) = \max_{s \in [0, +\infty)} \text{rank} \left(\Psi(s) + G \right).$$

Case 1: $r = m$. There exist $1 \leq j_1 < j_2 < \dots < j_r \leq k$ such that the matrix $\tilde{\Psi}(s)$ composed by j_i -th rows ($i = 1, 2, \dots, r$) of $\Psi(s) + G$ is invertible at $s = s_0$. Then since every element in $\tilde{\Psi}(\cdot)$ are rational functions, the determinant of $\tilde{\Psi}(\cdot)$ is a rational function and is not identical to zero. Consequently, $\tilde{\Psi}(s)$ is invertible on $[0, +\infty)$ except for finite points. Let $\tilde{\Phi}(s)$ be the matrix composed by j_i -th ($i = 1, 2, \dots, r$) rows of $\Phi(s)$. Then

$$U(s) = \tilde{\Psi}(s)^{-1} \tilde{\Phi}(s)X_0, \quad s \in [0, +\infty).$$

Thus, every element in $U(s)$ are rational functions. Moreover, noting that the inverse Laplace transform of a (non-zero) polynomial is the linear combination of δ function and its derivatives, elements in $U(s)$ must be proper rational functions since $u(\cdot)$ is analytic. Thus, $u(\cdot)$ has the form (3.3). Therefore, in this case, we can get our result by choosing $\overline{X}_0 = X_0$ and $\bar{u}(\cdot) = u(\cdot)$.

Case 2: $r < m$. Let $\overline{X}_0 = 0$. We will prove that there exists a $\bar{u}(\cdot) \neq 0$ such that $(\overline{X}_0, \bar{u}(\cdot))$ satisfies (3.2) and (3.5).

Let $1 \leq j_1 < j_2 < \dots < j_r \leq k$ satisfy that the matrix $\tilde{\Psi}(s)$ composed by j_i -th rows ($i = 1, 2, \dots, r$) of $\Psi(s) + G$ has full row rank when $s = s_0$. Then $\tilde{\Psi}(s)$ has full row rank except for finite points. Moreover, the equation

$$(\Psi(s) + G)V(s) = 0 \quad (3.7)$$

is equivalent to

$$\tilde{\Psi}(s)V(s) = 0. \quad (3.8)$$

Without loss of generality, suppose that

$$\tilde{\Psi}(s) = \begin{pmatrix} \tilde{\Psi}_1(s) & \tilde{\Psi}_2(s) \end{pmatrix},$$

where $\tilde{\Psi}_1(s)$ is an $r \times r$ matrix-valued function such that $\tilde{\Psi}_1(s)$ is invertible at $s = s_0$. Then since elements in $\tilde{\Psi}_1(s)$ are rational functions, $\tilde{\Psi}_1(s)$ is invertible except for finite points. Consequently, it is easy to see that (3.8) admits a solution $V_1(\cdot)$ with

$$V_1(s) = \begin{pmatrix} Q_1(s) \\ \vdots \\ Q_r(s) \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

where $Q_j(s)$ ($j = 1, \dots, r$) are rational functions. Choosing J large enough and letting

$$V(s) = \frac{V_1(s)}{s^J},$$

we get a nontrivial solution $V(\cdot)$ of (3.8)(or (3.7), equivalently) such that every element of $V(\cdot)$ are proper rational functions. Consequently,

$$\bar{u}(\cdot) = \mathcal{L}^{-1}(V(\cdot)) \neq 0$$

is well-defined and $\bar{u}(\cdot)$ has the form (3.5). Moreover,

$$C \int_0^t e^{A(t-s)} B \bar{u}(s) ds + G \bar{u}(t) = 0, \quad \forall t \geq 0.$$

That is $(\bar{X}_0, \bar{u}(\cdot))$ satisfies (3.2).

We get the proof. □

Using the properties of the differential operator D , we can go further.

Lemma 3.2 *If S_1 and S_2 are not distinguishable, then we can find a pair $(\tilde{X}_0, \tilde{u}(\cdot)) \neq 0$ satisfying (3.2) and*

$$\tilde{u}(\cdot) = e^{\lambda t} \xi, \tag{3.9}$$

where $\lambda \in \mathbb{C}$ and $\xi \in \mathbb{C}^m$.

Proof. By Lemma 3.1, there exists a pair $(\bar{X}_0, \bar{u}(\cdot))$ satisfying (3.1)—(3.2) and (3.5).

Case 1: $\bar{u}(\cdot) \equiv 0$. Then let $\tilde{u}(\cdot) = \bar{u}(\cdot)$, we get the conclusion.

Case 2: $\bar{u}(\cdot) \not\equiv 0$. Then

$$\bar{u}(t) = e^{\lambda_1 t} P_1(t) + e^{\lambda_2 t} P_2(t) + \dots + e^{\lambda_q t} P_q(t),$$

where

$$P_i(t) = \xi_{p_i, i} t^{p_i} + \xi_{p_i-1, i} t^{p_i-1} + \dots + \xi_{1, i} t + \xi_{0, i}, \quad i = 1, \dots, q.$$

$$p_i \geq 0, \quad \xi_{p_i, i} \neq 0,$$

and

$$\lambda_i \neq \lambda_j, \quad i \neq j.$$

Let $(X(\cdot), Y(\cdot))$ be the solution of (1.5) corresponding to $(\bar{X}_0, \bar{u}(\cdot))$. Then

$$Y(t) \equiv CX(t) + G\bar{u}(t) \equiv 0$$

since $(\bar{X}_0, \bar{u}(\cdot))$ satisfies (3.2).

Let

$$Q(\lambda) = (\lambda - \lambda_1)^{p_1} (\lambda - \lambda_2)^{p_2+1} \dots (\lambda - \lambda_q)^{p_q+1},$$

$$\tilde{u}(t) = Q(D)\bar{u}(t)$$

and

$$\tilde{X}(t) = Q(D)X(t), \quad \tilde{Y}(t) = Q(D)Y(t).$$

We have $\tilde{Y} \equiv 0$,

$$\frac{d\tilde{X}}{dt} = A\tilde{X}(t) + B\tilde{u}(t),$$

and

$$\tilde{Y}(t) = C\tilde{X}(t) + G\tilde{u}(t).$$

That is, $(\tilde{X}(\cdot), \tilde{Y}(\cdot))$ is the solution of (1.5) corresponding to $(\tilde{X}_0, \tilde{u}(\cdot))$ for some $\tilde{X}_0 \in \mathbb{R}^{2n}$. In other words, $(\tilde{X}_0, \tilde{u}(\cdot))$ satisfies (3.2).

Finally, (3.1) follows from

$$\begin{aligned} \tilde{u}(t) &= Q(D)\bar{u}(t) \\ &= \sum_{i=1}^q e^{\lambda_i t} Q(D + \lambda_i) P_i(t) \\ &= e^{\lambda_1 t} Q(D + \lambda_1) P_1(t) \\ &= e^{\lambda_1 t} (D + \lambda_1 - \lambda_2)^{p_2+1} \dots (D + \lambda_1 - \lambda_q)^{p_q+1} D^{p_1} P_1(t) \\ &= (p_1)! (\lambda_1 - \lambda_2)^{p_2+1} \dots (\lambda_1 - \lambda_q)^{p_q+1} e^{\lambda_1 t} \xi_{p_1, 1} \\ &\neq 0. \end{aligned}$$

□

Corollary 3.3 *If S_1 and S_2 are 0-th polynomial input distinguishable, then they are polynomial input distinguishable.*

Proof. Subsystems S_1 and S_2 are 0-th polynomial input distinguishable means that for any $(x_{10}, x_{20}, u(\cdot)) \in R^n \times R^n \times \mathcal{U}$ with $u(\cdot) \equiv \xi \in \mathbb{R}^m$, the outputs $y_1(\cdot)$ and $y_2(\cdot)$ can not be identical to each other on $[0, T]$.

If S_1 and S_2 are not polynomial input distinguishable, then there exists $(X_0, u(\cdot)) \neq 0$ such that (3.2) holds with $u(\cdot)$ being a polynomial. Then using the method we constructed $\tilde{u}(\cdot)$ from $\bar{u}(\cdot)$ in the proof of Lemma 3.2, we can construct a pair $(\tilde{X}_0, \tilde{u}(\cdot)) \neq 0$ satisfying (3.2) with $\tilde{u}(\cdot)$ being a constant vector. This means S_1 and S_2 are not 0-th polynomial input distinguishable. \square

By Corollary 3.3, the necessary and sufficient condition for 0-th polynomial input distinguishable and that for k -th polynomial input distinguishable are equivalent. Thus, by Theorem 3.1 of [8], we can see that for any $p \geq 0$, the matrix

$$\begin{pmatrix} C & G & 0 & \dots & 0 \\ CA & CB & G & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{p+2} & CA^{p+1}B & CA^pB & \dots & G \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

has full column rank if and only if

$$\begin{pmatrix} C & G \\ CA & CB \\ CA^2 & CAB \\ CA^3 & CA^2B \\ \vdots & \vdots \end{pmatrix}$$

has full column rank. Thus, it follows from Cayley-Hamilton's theorem, they are both equivalent to that

$$\begin{pmatrix} C & G \\ CA & CB \\ CA^2 & CAB \\ \vdots & \vdots \\ CA^{2n} & CA^{2n-1}B \end{pmatrix}$$

has full column rank.

Now, we state our main result.

Theorem 3.4 *Systems S_1 and S_2 are distinguishable if and only for any $\lambda \in \mathbb{C}$,*

$$\mathcal{M}_\lambda \equiv \begin{pmatrix} C & G \\ C(A - \lambda I) & CB \\ C(A - \lambda I)^2 & C(A - \lambda I)B \\ \vdots & \vdots \\ C(A - \lambda I)^{2n} & C(A - \lambda I)^{2n-1}B \end{pmatrix} \quad (3.10)$$

has full column rank.

Proof. (i) Suppose that S_1 and S_2 are distinguishable. Let $\lambda \in \mathbb{C}$. Consider

$$\begin{cases} \frac{d\tilde{X}(t)}{dt} = (A - \lambda I)\tilde{X}(t) + B\tilde{u}(t), \\ \tilde{X}(0) = \tilde{X}_0, \\ \tilde{Y}(t) = C\tilde{X}(t) + G\tilde{u}(t). \end{cases} \quad (3.11)$$

We claim for any $\tilde{x} \in \mathbb{C}^{2n}$ and $\xi \in \mathbb{C}^m$, $(\tilde{X}_0, \xi) \neq 0$, the solution of (3.11) corresponding to \tilde{X}_0 and

$$\tilde{u}(t) \equiv \xi$$

satisfies

$$\tilde{Y}(t) \not\equiv 0, \quad \text{on } [0, +\infty).$$

In other words, $(A_1 - \lambda I, B_1, C_1, G_1)$ and $(A_2 - \lambda I, B_2, C_2, G_2)$ are 0-th polynomial input distinguishable¹.

If it is not the case, then we have $(\tilde{X}_0, \xi) \neq 0$ such that the corresponding $\tilde{Y}(\cdot)$ equals to zero identically.

Let

$$X(t) = e^{\lambda t} \tilde{X}(t), \quad Y(t) = e^{\lambda t} \tilde{Y}(t),$$

Then $(X(\cdot), Y(\cdot))$ solves (1.5) with

$$X_0 = \tilde{X}_0, \quad u(t) = e^{\lambda t} \tilde{u}(t).$$

Since

$$Y(t) = e^{\lambda t} \tilde{Y}(t) = 0,$$

by considering the real part or imaginary part of $X_0, u(\cdot), X(\cdot)$ and $Y(\cdot)$, one can easily see that S_1 and S_2 are not distinguishable. This is a contradiction.

Similar to Theorem 3.1 of [8], we can get that the 0-th polynomial input distinguishable of $(A_1 - \lambda I, B_1, C_1, G_1)$ and $(A_2 - \lambda I, B_2, C_2, G_2)$ (in complex variable sense) implies that \mathcal{M} has full column rank.

(ii) Suppose that S_1 and S_2 are not distinguishable. Then, Lemma 3.2 shows that there is a pair $(X_0, u(\cdot)) \neq 0$ satisfying (3.2) and

$$u(\cdot) = e^{\lambda t} \xi, \quad (3.12)$$

for some $\lambda \in \mathbb{C}$. This implies that $(A_1 - \lambda I, B_1, C_1, G_1)$ and $(A_2 - \lambda I, B_2, C_2, G_2)$ are not 0-th polynomial input distinguishable. Consequently, \mathcal{M}_λ has not full column rank. \square

¹Here variables are complex, but this can be treated similarly to the case of that only real variables are concerned.

4 Generalization

In §1, the state variables are taken values in \mathbf{R}^n . In fact, we can consider more general cases. That is, for subsystem S_i of (1.1), suppose that

$$A_i \in \mathbf{R}^{n_i \times n_i}, \quad B_i \in \mathbf{R}^{n_i \times m}, \quad C_i \in \mathbf{R}^{k \times n_i}, \quad G_i \in \mathbf{R}^{k \times m}, \quad i = 1, 2,$$

where $n_1, n_2, k, m \geq 1$.

Similar to Definition 1.1, we define

Definition 4.1 *Systems S_1 and S_2 are said to be **distinguishable** on $[0, T]$, if for any non-zero*

$$(x_{10}, x_{20}, u(\cdot)) \in \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \times L^1(0, T; \mathbf{R}^m),$$

the corresponding output $y_1(\cdot; x_{10}, u(\cdot))$ of S_1 (satisfying the initial condition $x(0) = x_{10}$) and $y_2(\cdot; x_{20}, u(\cdot))$ of S_2 (satisfying the initial condition $x(0) = x_{20}$) are not identical to each other on $[0, T]$.

We have

Theorem 4.2 *Subsystems S_1 and S_2 are distinguishable if and only if for any $\lambda \in \mathbb{C}$, the matrix*

$$\mathcal{M}_\lambda \equiv \begin{pmatrix} C & G \\ C(A - \lambda I) & CB \\ C(A - \lambda I)^2 & C(A - \lambda I)B \\ \vdots & \vdots \\ C(A - \lambda I)^{n_1+n_2} & C(A - \lambda I)^{n_1+n_2-1}B \end{pmatrix} \quad (4.1)$$

has full column rank, where

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \in \mathbf{R}^{(n_1+n_2) \times (n_1+n_2)}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \in \mathbf{R}^{(n_1+n_2) \times m},$$

$$C = (C_1 \quad -C_2) \in \mathbf{R}^{k \times (n_1+n_2)}, \quad G = G_1 - G_2 \in \mathbf{R}^{k \times m}.$$

At the end of the paper, we prove the following equivalent result.

Theorem 4.3 *Let \mathcal{M}_λ be defined by (4.1). Then \mathcal{M}_λ has full column rank or any $\lambda \in \mathbb{C}$ if and only if for any $\lambda \in \mathbb{C}$*

$$\begin{pmatrix} C & G \\ A - \lambda I & B \end{pmatrix}$$

has full column rank.

Proof. Let $\tilde{C} = (C \ G)$, $\tilde{A}_\lambda = \begin{pmatrix} A - \lambda I & B \\ 0 & 0 \end{pmatrix}$. Then it is well known that for any $\lambda \in \mathbb{C}$,

$$\begin{pmatrix} \tilde{C} \\ \tilde{C}\tilde{A}_\lambda \\ \vdots \\ \tilde{C}\tilde{A}_\lambda^{n_1+n_2+m-1} \end{pmatrix}$$

has full column rank if and only if for any $\lambda, \mu \in \mathbb{C}$, $\begin{pmatrix} \tilde{C} \\ \tilde{A}_\lambda - \mu I \end{pmatrix}$ has full column rank (see [12] for example).

Thus, using Cayley-Hamilton's Theorem, for any $\lambda \in \mathbb{C}$, \mathcal{M}_λ has full column rank if and only if for any $\lambda, \mu \in \mathbb{C}$,

$$\begin{pmatrix} C & G \\ A - \lambda I & B \\ 0 & -\mu I \end{pmatrix}$$

has full column rank. While the later is equivalent to that for any $\lambda \in \mathbb{C}$, $\begin{pmatrix} C & G \\ A - \lambda I & B \end{pmatrix}$ has full column rank. We get the proof. \square

References

- [1] Balluchi, A., Benvenuti, L., DiBenedetto, M. D. and Sangiovanni-Vincentelli, A. L., Observability for Hybrid Systems. Proceedings of the 42nd IEEE Conference on Decision and Control, Maui, Hawaii, 2003, 1159 – 1164.
- [2] Babaali, M., Egerstedt, M., Observability for Switched Linear Systems. In: Hybrid Systems: Computation and Control, Lecture Notes in Computer Science, Vol. 2993, Springer-Verlag, 2004, 48 – 63.
- [3] Babaali, M. and Egerstedt, M., On the Observability of Piecewise Linear Systems, Proceedings of the 43rd IEEE Conference on Decision and Control, Vol. 1, Atlantis, Paradise Island, Bahamas, 2004, 26 – 31.
- [4] Bemporad, A., Ferrari-Trecate, G. and Morari, M., Observability and Controllability of Piecewise Affine and Hybrid Systems. *IEEE Transactions on Automatic Control*, 45(10), 1864 – 1876(2000).
- [5] Babaali, M. and Pappas, G. J., Observability of Switched Linear Systems in Continuous Time. In: Hybrid Systems: Computation and Control 2005, Lecture Notes in Computer Science, Vol. 3414, Springer-Verlag, 2005, 103 – 117.
- [6] Collins, P. and Schuppen, Jan H. van, Observability of piecewise-affine hybrid systems. In: Hybrid Systems: Computation and Control, Lecture Notes in Computer Science, Vol. 2993, Springer, 2004, 265 – 279.
- [7] Ferrari-Trecate, G. and Gati, M., Computation observability regions for discrete-time hybrid systems. Proceedings of 42nd IEEE Conference on Decision and Control, Vol. 2, Maui, Hawaii, 2003, 1153 – 1158.

- [8] Lou, H. and Si, P., The Distinguishability of Linear Control Systems. *Nonlinear Analysis: Hybrid Systems*, 3(1), 21 – 38 (2009).
- [9] Oishi, M., Hwang, I. and Tomlin, C., Immediate observability of discrete event systems with application to user-interface design. Proceedings of IEEE Conference on Decision and Control, Vol. 3, Maui, Hawaii, 2003, 2665 – 2672.
- [10] Özveren, C. M. and Willsky, A. S., Observability of Discrete Event Dynamic Systems, *IEEE Transactions on Automatic Control*, 35(7), 979 – 806(1990).
- [11] Santis, E. De, DiBenedetto, M. D. and Pola, G., On observability and detectability of continuous-time linear switching systems. Proceedings of the 42nd IEEE Conference on Decision and Control, Vol. 6, Maui, Hawaii, 2003, 5777 – 5782.
- [12] Schrader, C. B. and Sain, M. K., Research on system zeros: A survey. *International Journal of Control*, 50(4), 1407 – 1433(1989).
- [13] Vidal, R., Chiuso, A. and Soatto, S., Observability and Identifiability of Jump Linear Systems. Proceedings of IEEE Conference on Decision and Control, Vol. 4, Las Vegas NV, 2002, 3614 – 3619.
- [14] Vidal, R., Chiuso, A., Soatto, S. and Sastry, S., Observability of Linear Hybrid Systems. In: *Hybrid Systems: Computation and Control*, Lecture Notes in Computer Science, Vol. 2623, Springer-Verlag, 2003, 526 – 539.